RUBIO DE FRANCIA LITTLEWOOD PALEY INEQUALITIES AND DIRECTIONAL MAXIMAL FUNCTIONS

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ABSTRACT. In $\mathbb{R}^d$, define a maximal function in the directions $v \in \mathsf{drns} \subset \{x : |x| = 1\}$ by

$$M^{\mathsf{drns}}f(x) = \sup_{v \in \mathsf{drns}} \sup_{\varepsilon} \int_{-\epsilon}^{\epsilon} |f(x - vy)|\, dy.$$

For a function f on $\mathbb{R}^d$, let $S_\omega f$ denote the Fourier restriction of f to a region ω. We are especially interested taking ω to be a sector of $\mathbb{R}^d$ with base points at the origin. A sector is a product of the interval $(0, \infty)$ with respect to a choice of (non orthogonal) basis. What is most important is that the basis is a subset of drns. Consider a collection Ω of pairwise disjoint sectors ω as above. Assume that M^{drns} maps L^p into L^p, for some $1 < p < \infty$. Then we have the following Littlewood–Paley inequality

$$\left\| \left[\sum_{\omega \in \Omega} |S_\omega f|^2 \right]^{1/2} \right\|_q \lesssim \|f\|_q, \qquad 2 \le q < 2\frac{p}{p-1}.$$

The one dimensional analogue of this inequality is due to Rubio de Francia, [10]. The conclusion when the set of vectors is a fixed basis is known, is due to Journé [6]. Our method of proof relies on a phase plane analysis. We introduce a notion of Carleson measures adapted to drns, and demonstrate a John Nirenberg inequality for these measures. The John Nirenberg inequality, and an obvious L^2 estimate will prove the Theorem.

1. INTRODUCTION

We are interested in the connection between Littlewood–Paley inequalities in higher dimensions, especially into parallelepipeds and sectors with respect to a variety of distinct bases. We demonstrate that the maximal function bounds imply Littlewood–Paley inequalities.

The classical Littlewood–Paley inequalities concern the decomposition of the frequency variables into lacunary pieces. Our subject is the extent to which these inequalities can be generalized when the decompositions of frequency variables are liberalized. We specifically generalize the beautiful inequality of Rubio de Francia [10] to the higher dimensions, namely decompositions of frequency variables are specified by an arbitrary collection of pairwise disjoint parallelepipeds and sectors. The paper concludes with several remarks about our Theorem, its relationship to prior work and possible generalizations.

In this paper, ω will denote a parallelepiped in $\mathbb{R}^d$. A parallelepiped is a product of intervals in a choice of (non orthogonal) coordinate axes. The axes, in particular, may may vary depending on the parallelepiped. If the intervals in question are $(0, \infty)$, so that the parallelepiped has a single vertex at the origin, then we say that it is a sector.

Define the Fourier restriction operator to be

$$S_\omega f(x) = \mathcal{F}^{-1} \mathbf{1}_\omega \mathcal{F} f(x),$$

where f is a function on the plane and $\mathcal{F} f(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) \, dx$ is the Fourier transform. For a collection of parallelepipeds Ω set

$$S^\Omega f = \left[\sum_{\omega \in \Omega} |S_\omega f|^2 \right]^{1/2}.$$

Let $\mathsf{drns} \subset \{x : |x| = 1\}$ be a set of norm one vectors in $\mathbb{R}^d$.

$$M^{\mathsf{drns}} f(x) = \sup_{R \in \mathcal{P}^{\mathsf{drns}}} \mathbf{1}_R(x) \fint_R |f(y)| \, dy$$

1.1. Theorem. *Let Ω be any collection of disjoint parallelepipeds, and assume that each element of Ω is a parallelepiped with respect to a basis drawn from vectors in drns. Assume that M^{drns} maps L^p into L^p, for some $1 < p < \infty$. Then the square function S^Ω maps $L^q(\mathbb{R}^d)$ into itself for $2 \le q < 2\frac{p}{p-1} = 2p'$. More particularly, for a choice of constant $\kappa = \kappa(p, d)$,*

$$(1.2) \qquad \|S^\Omega\|_q \lesssim \|M^{\mathsf{drns}}\|_{p \to p}^\kappa, \qquad 2 \le q < 2\frac{p}{p-1} = 2p$$

Notice that a sector is an increasing limit of parallelepipeds so that the inequality stated in the abstract is an immediate consequence of the Theorem.

One would not suppose that the method adopted here would supply an optimal estimate for κ in (1.2).

We adopt a method of proof that emphasizes a notion of space–frequency analysis, as following the notes of [7]. That paper concentrates on rectangles with respect to a fixed set of coordinate axes, and the product BMO theory of Chang and R. Fefferman, [2, 3]. More exactly, that paper highlights the role of the product Carleson measures in that the case of rectangles with respect to a fixed set of coordinate axes.

For our current theorem, clearly there is no such theory, and so we must find appropriate analogues in this setting.

The one dimensional version of this result is the striking result of Rubio de Francia [10]. The two dimensional version, with parallelepipeds with respect to a fixed choice of axes, was

proved by Journé [6]. Several other authors have made contributions in this direction, we cite, without further comment: Bourgain [1], Córdoba [5], Olevskii [8, 9], Sato [11], Sjölin [11], and Zhu [13]. These issues are surveyed in a recent article by one of us [7]. In particular, the view point we take is heavily influenced by that survey article.

Our theorem, in the case of the plane, and uniformly distributed sectors is due to A. Cordoba [4].

2. Reduction to the Well Distributed Case

This section follows the reduction used by Rubio de Francia [10] to collections of intervals that are better suited to frequency analysis. Let Ω be a collection of disjoint parallelepipeds ω, each a parallelepiped with respect to a choice of basis from drns. Assume in addition that Ω satisfies

$$(2.1) \qquad \left\| \sum_{\omega \in \Omega} \mathbf{1}_{2\omega} \right\|_{\infty} < \infty.$$

We say that Ω is *well distributed.*

2.2. Lemma. *For any collection Ω, we can select a well distributed collection* $\mathrm{Well}(\Omega)$ *for which we have the inequality*

$$\|S^{\Omega}\|_q \lesssim \|M^{\mathsf{drns}}\|_{(q/2)'}^{\kappa} \|S^{\mathrm{Well}(\Omega)}\|_q, \qquad 1 < q < \infty.$$

In this Lemma, and throughout this paper, κ denotes a positive number, whose exact value we shall not attempt to keep track of.

Consider first the one dimensional case, as we shall be able to pass to higher dimensions by taking appropriate products. In turn, in one dimension, we first consider the interval $[-\frac{1}{2}, \frac{1}{2}]$. Set

$$\mathrm{Well}([-\tfrac{1}{2}, \tfrac{1}{2}]) = \{[-\tfrac{1}{18}, \tfrac{1}{18}], \pm[\tfrac{1}{2} - \tfrac{4}{9}(\tfrac{4}{5})^k, \tfrac{1}{2} - \tfrac{4}{9}(\tfrac{4}{5})^{k+1}] : k \geq 0\}.$$

It is straightforward to check that all the intervals in this collection have a distance to the boundary of $[-\frac{1}{2}, \frac{1}{2}]$ that is four times their length. In particular, this collection is well distributed, and for each $\omega \in \mathrm{Well}([-\frac{1}{2}, \frac{1}{2}])$ we have $2\omega \subset [-\frac{1}{2}, \frac{1}{2}]$.

We define $\mathrm{Well}(\omega)$ by affine invariance. For an interval ω, select an affine function $\alpha : [-\frac{1}{2}, \frac{1}{2}] \longrightarrow \omega$, we set $\mathrm{Well}(\omega) := \alpha(\mathrm{Well}([-\frac{1}{2}, \frac{1}{2}]))$. And we define $\mathrm{Well}(\Omega) := \bigcup_{\omega \in \Omega} \mathrm{Well}(\omega)$. It is clear that $\mathrm{Well}(\Omega)$ is well distributed for collections of disjoint intervals Ω.

We shall estimate the the L^q norms of these square functions by duality, as we are always interested in $q > 2$. There is a standard weighted inequality, valid for all $\epsilon > 0$, that we shall

appeal to several times.

$$(2.3) \qquad \int_{\mathbb{R}} |S_\omega f|^2 g \, dx \lesssim \int_{\mathbb{R}} |S^{\text{Well}(\omega)} f|^2 (M|g|^{1+\epsilon})^{1/1+\epsilon} \, dx, \qquad \epsilon > 0.$$

Here, M is the one dimensional maximal function. Clearly, this extends immediately to the collection of intervals Ω, and the square function S^Ω.

For the case of a parallelepiped ω, we write it as a product of intervals $\omega = \prod_{j=1}^{d} \omega_j$, in the appropriate non orthogonal coordinates. We then define $\text{Well}(\omega) = \prod_{j=1}^{d} \text{Well}(\omega)_j$. By iterating (2.3) in each coordinate, we see that

$$(2.4) \qquad \int_{\mathbb{R}^d} |S_\omega f|^2 g \, dx \lesssim \int_{\mathbb{R}^d} |S^{\text{Well}(\omega)} f|^2 (M^\omega |g|^{1+\epsilon})^{1/1+\epsilon} \, dx, \qquad 1 < p < \infty.$$

in which M^ω is a d times iterate of one dimensional maximal functions in the coordinates associated to ω. This Lemma can be summed over a collection of parallelepipeds Ω, with the change that the maximal function M^{drns}, iterated d times, must be imposed on the right hand side of the inequality. Thus the Lemma is proved.

We use the notion of well distributed just as Rubio de Francia did, to pass from the sharp Fourier restriction, with it's accompanied long range spatial behavior, to convolution with Schwartz functions, with very rapid spatial decay.

2.5. Lemma. *We assume that Ω is a set of parallelepipeds in the set of directions* drns. *Assume that M^{drns} is bounded on L^p. Let ψ_ω, for $\omega \in \Omega$, be a Schwartz function with $\mathbf{1}_\omega \leq \widehat{\psi_\omega} \leq \mathbf{1}_{2\omega}$. Then,*

$$\|S^\Omega f\|_q \lesssim \|M^{\text{drns}}\|_{p \to p}^\kappa \left\| \left[\sum_{\omega \in \Omega} |\psi_\omega * f|^2 \right]^{1/2} \right\|_q, \qquad 2 < q < 2p'.$$

The proof is again an application of (2.4), in exactly the same manner.

3. The Space–Frequency Tiles

Our purpose is to define a discrete analog of the square function S^Ω. The discrete analog will more easily permit an analysis in terms of the theory of Carleson measures in directions that we develop in Section 5 and the subsequent sections.

Let us introduce the operators associated to translation, modulation and dilation.

$$(3.1) \qquad\qquad \text{Tr}_y f(x) = f(x - y), \quad y \in \mathbb{R}^d$$

$$(3.2) \qquad\qquad \text{Mod}_\xi f(x) = e^{i\xi \cdot x} f(x), \quad \xi \in \mathbb{R}^d.$$

A parallelepiped $R \subset \mathbb{R}^d$ is a product of intervals $R = \prod_{i=1}^d R_i$ with respect to a choice of basis in $\mathbb{R}^d$, with coordinates $(x_1, x_2, \ldots, x_d)$. Set a dilation operator associated to R to be

$$(3.3) \qquad \operatorname{Dil}_R^p f(x) = |R|^{-1/p} f(\alpha(x)), \qquad 0 \leq p < \infty,$$

Here, α is an affine map that carries $R = \prod_{i=1}^d R_i$ coordinate wise into the standard cube $[-1/2, 1/2]^d$. This depends on the orientation of R, and its side lengths of R, and its location. Notice that this definition implictly incorporates a translation.

Two parallelepipeds ω and R are said to be *dual* if they are both parallelepipeds with respect to the same choice of coordinate vectors, and with those vectors ordered, writing $\omega = \prod_{j=1}^d \omega_j$ and $R = \prod_{j=1}^d R_j$, one has

$$(3.4) \qquad 1 \leq |R_j| \cdot |\omega_j| \leq 2, \qquad 1 \leq j \leq d.$$

Let $\mathcal{D}(\mathsf{drns})$ denote the collection of parallelepipeds that are dyadic with respect to a choice of basis in drns. That is, the parallelepiped is a product of dyadic intervals in it's basis.[1]

Call a product $R \times \omega$ a *tile* if ω and R are dual, and $R \in \mathcal{D}(\mathsf{drns})$. We shall associate to each tile an appropriate function. Fix a Schwartz function $\varphi \geq 0$ with $\widehat{\varphi}$ supported on $[-9/16, 9/16]^d$, set

$$(3.5) \qquad \varphi_{R \times \omega} = \operatorname{Mod}_{c(\omega)} \operatorname{Dil}_R^2 \varphi$$

where $c(J)$ denotes the center of J.

Let $\mathcal{T}(\Omega)$ be a collection of tiles so that for all $s = R_s \times \omega_s \in \mathcal{T}(\Omega)$, we have that $\omega_s \in \Omega$. Our space–frequency square function is

$$(3.6) \qquad SF^\Omega f = \Big[\sum_{s \in \mathcal{T}(\Omega)} \frac{|\langle f, \varphi_s \rangle|^2}{|R_s|} \mathbf{1}_{R_s} \Big]^{1/2}.$$

We let drns be the set of coordinates for the parallelepipeds in Ω.

3.7. Proposition. *Assume that M^{drns} maps L^p into L^p for some $1 < p < \infty$. Then, for all well distributed Ω, we have*

$$\| SF^\Omega \|_q \lesssim \| M^{\mathsf{drns}} \|_p^\kappa, \qquad 2 \leq q < 2p'.$$

We impose the well distributed assumption to trivialize the boundedness of the square function on L^2. Indeed, by our construction, the function φ_s is supported on $2\omega_s$, and these sets have bounded overlap. Thus, the L^2 inequality reduces to checking it for Ω consisting of just one parallelepiped. But then, the fact that φ is a Schwartz function proves the desired inequality.

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[1]This restriction is made to keep $\mathcal{D}(\mathsf{drns})$ countable. Our Carleson measure theory can then be phrased in terms of sums, rather than integrals.

The main point in the proof of the proposition is this fact, which we state in the language of Section 5.

3.8. Lemma. *Fix $\epsilon > 0$. Assume that M^{drns} maps L^p into L^p for some $1 < p < \infty$, and that Ω is well distributed. Then for each f on $\mathbb{R}^d$, with L^∞ norm bounded by one, the map from $\mathcal{D}(\mathsf{drns})$ to $\mathbb{R}_+$ below has $CM(\mathsf{drns})$ norm at most $\|M^{\mathsf{drns}}\|_{p \to p}^\kappa$.*

$$\mathcal{D}(\mathsf{drns}) \ni R \longrightarrow \sum_{\substack{s \in \mathcal{T}(\Omega) \\ R_s = R}} |\langle f, \varphi_s \rangle|^2$$

4. Proofs

Proof of Proposition 3.7. Let $q = 2\frac{p}{p-1} = 2p'$. As the L^2 inequality is a consequence of the well distributed assumption, we need only show the restricted weak type inequality at L^q. Fix a function $f = \mathbf{1}_F$, where F is of finite measure. For a set of of tiles $\mathcal{S} \subset \mathcal{T}(\Omega)$, let

$$S(\mathcal{S}) := \sum_{s \in \mathcal{S}} \frac{|\langle f, \varphi_s \rangle|^2}{|R_s|} \mathbf{1}_{R_s}$$

[Note the lack of a square root here.] It is our task to show that for all $\lambda > 0$,

(4.1) $$|\{S(\mathcal{T}(\Omega)) > \lambda\}| \lesssim \lambda^{-p'} \|M^{\mathsf{drns}}\|_p^\kappa |F|.$$

As the admissible collections of tiles are invariant under dilations which are uniform in all coordinates, it suffices to consider the case of $\lambda = 1$ in this inequality.

In addition define

$$\mathrm{sh}(\mathcal{S}) := \bigcup_{s \in \mathcal{S}} R_s.$$

$$\mathrm{size}(\mathcal{S}) := \sup_{\mathcal{S}' \subset \mathcal{S}} \left[|\mathrm{sh}(\mathcal{S}')|^{-1} \sum_{s \in \mathcal{S}'} |\langle f, \varphi_s \rangle|^2 \right]^{1/2}$$

The definition of size is equivalent to that of a Carleson measure. It follows immediately from Lemma 3.8 that we have $\mathrm{size}(\mathcal{T}(\Omega)) \leq \mu \lesssim \|M^{\mathsf{drns}}\|_p^\kappa$. And moreover, from Lemma 5.2 we have the inequality of John Nirenberg type

(4.2) $$\|S(\mathcal{S})\|_{p'} \lesssim \mathrm{size}(\mathcal{S})^2 |\mathrm{sh}(\mathcal{S})|^{1/p'}.$$

The next stage of the argument is the primary decomposition of the set of tiles $\mathcal{T}(\Omega)$. Note that if there are two disjoint subcollections $\mathcal{S}^j$, $j = 1, 2$ of $\mathcal{T}(\Omega)$ such that for both we have

$$\sum_{s \in \mathcal{S}^j} |\langle \mathbf{1}_F, \varphi_s \rangle|^2 \geq \frac{1}{2} |\mathrm{sh}(\mathcal{S}^j)| \mu^2,$$

then the same inequality holds for their union. Thus there is a maximal (not necessarily unique) subcollection $\mathcal{S}_1 \subset \mathcal{T}(\Omega)$ such that

$$\sum_{s \in \mathcal{S}_1} |\langle \mathbf{1}_F, \varphi_s \rangle|^2 \geq \tfrac{1}{2} |\mathrm{sh}(\mathcal{S}_1)| \mu^2,$$

$$\mathrm{size}(\mathcal{T}(\Omega) - \mathcal{S}_1) \leq |\mathrm{sh}(\mathcal{S})| \mu.$$

The last part suggests a recursive application to $\mathcal{T}(\Omega) - \mathcal{S}_1$. Carrying this out will result in a decomposition of $\mathcal{T}(\Omega)$ into collections $\mathcal{S}_k$, $k \geq -2 \log \mu$, for which for each k, we have

(4.3)
$$\sum_{s \in \mathcal{S}_k} |\langle \mathbf{1}_F, \varphi_s \rangle|^2 \geq 2^{-2k} \mu^2 |\mathrm{sh}(\mathcal{S}_k)|,$$

$$\mathrm{size}(\mathcal{S}_k) \lesssim 2^{-k} \mu.$$

The top inequality gives an upper bound on $|\mathrm{sh}(\mathcal{S}_k)| \lesssim 2^{2k} \mu^2 |F|$. This follows from the L^2 inequality that holds by design.

Then, for a small value of ϵ, we will have $\sum_{k \geq -2 \log \mu} \epsilon 2^{-\epsilon k} \leq 1$, so that it suffices to see that

$$\sum_{k \geq -2 \log \mu} |\{ S(\mathcal{S}_k) > \epsilon 2^{-\epsilon k} \}| \lesssim \mu^\kappa |F|,$$

for some absolute choice of κ. We can employ the John Nirenberg inequality (4.2) to see that

$$|\{ S(\mathcal{S}_k) > \epsilon 2^{-\epsilon k} \}| \leq 2^{-k(-\epsilon)2p'} |\mathrm{sh}(\mathcal{S}_k)|$$
$$\lesssim 2^{-k[(1-\epsilon)2p'-2]} |F|.$$

This is summable in $k \geq -2 \log \mu$ to $\mu^\kappa |F|$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Lemma 3.8. The main points are the well distributed assumption, and the boundedness of the maximal function M^{drns}. Fix a function f on $\mathbb{R}^d$ that is bounded by one, and an open set $U \subset \mathbb{R}^d$. We are to show that

$$\sum_{\substack{s \in \mathcal{T}(\Omega) \\ R_s \subset U}} |\langle f, \varphi_s \rangle|^2 \lesssim \| M^{\mathsf{drns}} \|_{p \to p}^\kappa \| f \|_\infty^2 |U|.$$

We need this elementary Lemma.

4.4. **Lemma.** *For all $N \geq 1$, $0 < a < 1$, $\omega \in \Omega$ and functions f supported off of the set $\{ M^{\mathsf{drns}} \mathbf{1}_U > a^d \}$, we have the estimate*

$$\sum_{\substack{R_s \subset U \\ \omega_s = \omega}} |\langle f, \varphi_s \rangle|^2 \lesssim a^N \| f \|_2^2$$

Proof. Let $\mathcal{T}$ consist of those tiles $s \in \mathcal{T}(\Omega)$ for which $R_s \subset U$ and $\omega_s = \omega$. Let $\Gamma_s = \frac{1}{5}a^{-1}R_s$, and note that this parallelepiped cannot intersect the support of f. We decompose $\varphi_s = \alpha_s + \beta_s$, where α_s is a smooth function supported on Γ_s, and equal to φ_s on $\frac{1}{2}\Gamma_s$.

Thus, β_s is the "trivial" part. And indeed, it is straightforward to verify that

$$\sum_{s \in \mathcal{T}} |\langle g, \beta_s \rangle|^2 \lesssim a^N \|g\|_2^2.$$

But we will not apply this estimate to f, but rather to $\psi_{2\omega} * f$. Then, we have

$$\langle f, \varphi_s \rangle = \langle \varphi^{2\omega} * f, \varphi_s \rangle = \langle \varphi^{2\omega} * f, \alpha_s \rangle + \langle \varphi^{2\omega} * f, \beta_s \rangle$$

Certainly, we do not need to futher consider the inner products with β_s. As for the inner products with α_s, the main point is that $\varphi^{2\omega} * f$ is dominated by appropriate iterate of a maximal function M in coordinates for the parallelepiped ω. And in fact, using the assumption about the support of f, and the fact that φ is a Schwartz function, for each $s \in \mathcal{T}$, and $x \in \Gamma_s$, we have

$$|\varphi^{2\omega} * f(x)| \lesssim a^N M f(x).$$

Therefore, we can estimate by Cauchy–Schwarz,

$$\sum_{s \in \mathcal{T}} |\langle \varphi^{2\omega} * f, \alpha_s \rangle|^2 \lesssim a^N \sum_{s \in \mathcal{T}} \int_{\Gamma_s} |Mf|^2 \, dx$$

$$\lesssim a^{N-d} \|f\|_2^2.$$

The last line follows from the L^2 bound on M that is uniform in the choice of ω, and since the rectangles Γ_s can overlap at most $\lesssim a^{-d}$ times.

$\square$

Let us decompose f into a sum of functions f_k, for $k \geq 0$,

$$f_0 = f\mathbf{1}_{\{1/2 \leq M^{\mathsf{drns}}\mathbf{1}_U\}},$$

$$f_k = f\mathbf{1}_{\{2^{-k-1} \leq M^{\mathsf{drns}}\mathbf{1}_U \leq 2^{-k}\}}, \qquad k > 0.$$

For f_0, we use the well distributed assumption, and the Bessel inequality it implies, to see that

$$\sum_{\substack{s \in \mathcal{T}(\Omega) \\ R_s \subset U}} |\langle f_0, \varphi_s \rangle|^2 \lesssim \|f_0\|_2^2 \leq \|f\|_\infty^2 |U|$$

The terms f_k require Lemma 4.4. For each $\omega \in \Omega$,

$$\sum_{\substack{R_s \subset U \\ \omega_s = \omega}} |\langle f_k, \varphi_s \rangle|^2 \lesssim 2^{-kp} \|\psi_\omega * f_k\|_2^2$$

By the well distributed assumption, this estimate can be summed over ω, to acheive the bound

$$\sum_{R_s \subset U} |\langle f_k, \varphi_s \rangle|^2 \lesssim 2^{-kp} \|f_k\|_2^2$$

The last fact to be noted is that by the boundedness of the strong maximal function on L^p, we have that $\|f_k\|_2 \lesssim 2^{kp/2} \|f\|_\infty$. Therefore, this estimate can be summed over $k > 0$, to conclude the proof. $\qquad\square$

Proof of Theorem 1.1. We indicate the proof of our main Theorem. Recall from Lemma 2.2 that it suffices to consdier well distributed collections Ω, and so we should argue from Proposition 3.7, and construct a smooth square function, as in Lemma 2.5, for which we have the same norm inequalities as in Proposition 3.7. Namely, if the maximal function M^{drns} is bounded on L^p, we should obtain norm inequalities for the square function for $2 < q < 2p'$. The argument that we present here is comprised of standard lines of reasoning.

Let us set

$$B^\Omega f := \Big[\sum_{\omega \in \Omega} |B_\omega f|^2 \Big]^{1/2},$$

$$B_\omega f := \sum_{\substack{s \in \mathcal{T}(\Omega) \\ \omega_s = \omega}} \langle f, \varphi_s \rangle \varphi_s$$

We observe that B^Ω maps L^q into itself for $2 < q < 2p'$. Indeed, one has the inequality

$$B^\Omega f \lesssim \Big[\sum_{s \in \mathcal{T}(\Omega)} |\langle f, \varphi_s \rangle \varphi_s|^2 \Big]^{1/2}$$

$$\lesssim \Big[\sum_{s \in \mathcal{T}(\Omega)} \frac{|\langle f, \varphi_s \rangle \varphi_s|^2}{|R_s|} (M^{R_s} \mathbf{1}_{R_s})^2 \Big]^{1/2}.$$

Here, in the top line, we are using Cauchy—Schwarz and the rapid decay of the functions φ_s. Notice that the bottom line is very similar to the square function considered in Proposition 3.7. The only difference is the imposition of the maximal function on the indicator of R_s. In particular, M^{R_s} is the maximal function in the basis for the rectangle R_s.

There is a counterpart of (2.3) that applies to the maximal function. Namely, in one dimension,

$$\int |Mf|^2 g\, dx \lesssim \int |f| (M|g|^{1+\epsilon})^{1/1+\epsilon}\, dx, \qquad \epsilon > 0.$$

Applying this to the right hand side above, and using the hypothesis that $\|M^{\mathsf{drns}}\|_{p \to p}$ is finite, we see that B^Ω satisfies the claimed range of L^q inequalities.

From B^{Ω} we should pass to an operator which is a square function of convolution operators as in Lemma 2.5. It suffices to consider each parallelepiped $\omega \in \Omega$ individually. Consider the limit

$$C_\omega f := \lim_{Y \to \infty} \int \mathrm{Tr}_{-y}\, B_\omega\, \mathrm{Tr}_y\ \mu_Y(dy)$$

where μ_Y is normalized Lebesgue measure on the ball centered at the origin of radius Y. One sees that this limit exists for all Schwartz functions. Since B_ω is clearly a bounded operator on L^2, we conclude that C_ω is as well. It also commutes with all translation operators, by construction. Hence, it is a convolution operator. And one may check directly that $C_\omega f = \psi_\omega$, where

$$\psi_\omega(x) = \int_{\mathbb{R}^d} \varphi(x+y)\overline{\varphi_\omega(y)}\ dy$$

By construction, $\mathbf{1}_\omega \le \widehat{\psi_\omega} \le \mathbf{1}_{2\omega}$, so that we have constructed a smooth square function as in Lemma 2.5. The proof of the Theorem is complete. $\qquad\square$

5. Carleson Measures with Directions

We set out a theory of Carleson measures associated to sets of directions drns in $\mathbb{R}^d$. Recall that $\mathcal{D}(\mathsf{drns})$ denotes the set of parallelepipeds in $\mathbb{R}^d$ that are dyadic with respect to a choice of bases from drns. When drns is a single orthogonal basis, all of this reduces to the Carleson measure theory associated with product BMO.

For a function $\Lambda : \mathcal{D}(\mathsf{drns}) \longrightarrow \mathbb{R}_+$, we set

$$(5.1) \qquad \|\Lambda\|_{CM(\mathsf{drns})} := \sup_{U \subset \mathbb{R}^d} |U|^{-1} \sum_{R \subset U} \Lambda(R)$$

What is to be emphasized, is that the supremum is taken over all subsets of $\mathbb{R}^d$ of finite measure.

Despite the generality of these definitions, it does permit the development of a rudimentary theory. The first fact to note is an extension of the John Nirenberg inequality.

5.2. Lemma. *Assume that M^{drns} maps L^p into weak L^p. Then we have the inequality below, valid for all sets $U \subset \mathbb{R}^d$, of finite measure.*

$$\left\| \sum_{R \subset U} \frac{\Lambda(R)}{|R|} \mathbf{1}_R \right\|_q \lesssim \|M^{\mathsf{drns}}\|_{p \to p,\infty} \|\Lambda\|_{CM(\mathsf{drns})}, \qquad 1 \le q \le p'.$$

Proof. The argument of [3] for the John Nirenberg inequality needs only modest modifications in the present setting. Define

$$F_U = \sum_{R \subset U} \frac{\Lambda(R)}{|R|} \mathbf{1}_R$$

We want to show that $\|F_U\|_{p'} \lesssim \|M^{\mathsf{drns}}\|_{p\to p,\infty}|U|^{1/p'}$. This we shall do by showing that there is an open $V \subset \mathbb{R}^d$ so that $|V| < \frac{1}{2}|U|$ so that

$$\|F_U\|_{p'} \lesssim \|M^{\mathsf{drns}}\|_{p\to p,\infty}|U|^{1/p'} + \|F_V\|_{p'}$$

An inductive argument proves the desired inequality.

This is done by duality. Thus, choose $g \in L^p$ of norm one so that $\|F_U\|_{p'} = \langle F_U, g\rangle$. Then let $V = \{M^{\mathsf{drns}}g > c|U|^{-1/p}\}$. For appropriate constant $c \simeq \|M^{\mathsf{drns}}\|_{p\to p,\infty}$, the measure of V is at most half of the measure of U. Note that if $R \not\subset V$, then $\fint_R g\, dx < c|U|^{-1/p}$. Hence,

$$\begin{aligned}
\|F_U\|_{p'} &= \langle F_U, g\rangle \\
&= \sum_{\substack{R \subset U \\ R \not\subset V}} \Lambda(R)\fint_R g\, dx + \langle F_V, g\rangle \\
&\lesssim \|M^{\mathsf{drns}}\|_{p\to p,\infty}|U|^{1/p'} + \|F_V\|_{p'}
\end{aligned}$$

$\square$

6. Concluding Remarks

Our result is unsatisfying, as it does not give new examples of maximal functions M^{drns} for which there are L^p bounds. Indeed, the only instance in which we have a range of examples is the plane, and it is this setting that these results will likely find application.

It would also be of interest to know that the theorem is sharp as to the range of indicies that we prove the square function inequalities. We can show that it is sharp in one case, in that of uniformly distributed directions in the plane.

For a large integer N, let Γ be a collection of pairwise disjoint sectors in the plane, with vertexes at the origin, and opening angle $2\pi/N$. The maximal function we have associated to this square function is one over two sets of uniformly distributed directions in the plane. It is well known that this maximal function admits bounds that are logarithmic in N for $p \geq 2$, but the bound blows up as a power of N for $1 < p \leq 2$. We conclude that the square function S^{Γ} admits logarithmic bounds for $2 < q < 4$.

A simple example shows that this is the correct range of indicies for which one has such a bound. Let φ be a Schwartz function with $\widehat{\varphi}$ non negative, radial, rotationally symmetric, and supported in a small annulus about $|\xi| = 1$. For each $\gamma \in \Gamma$, it is routine to see that $|S_\gamma\varphi| \gtrsim N^{-1}\mathbf{1}_{R_\gamma}$, where R_γ is a $1 \times N$ rectangle, with center at the origin, and long direction oriented in the direction of the bisectrix of γ.

We therefore have

$$\|S^\Gamma \varphi\|_q^2 \gtrsim \big\| N^{-1} \sum_{\gamma \in \Gamma} \mathbf{1}_{R_\gamma} \big\|_{q/2}$$

$$\gtrsim N^{1-4/q}, \qquad 4 < q < \infty,$$

preventing the possibility of a meaningful result for the square function on L^q for $q > 4$.

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